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On Some Applications of Circular Coordinates.

By F. Franklin.

The interesting geometrical questions treated by Humbert in a recent number of this Journal (Vol. X, p. 258) may be investigated with advantage by the use of "circular coordinates." The theorems relating to the orientation of systems of lines given in the article just cited, and the more general theorems of the same nature due to Laguerre and Humbert (see Humbert, Sur le théorème d'Abel et quelques-unes de ses applications géométriques, *Liouville*, 1887, III, 327), present themselves at once; some of them may be stated in a way which suggests more readily certain interesting cases of the theorems; and there naturally arise also some slight additions to the theorems.

A second application of circular coordinates is made in this paper. Namely, it is obvious that when x, y are understood to be circular coordinates, the differential equation

$$\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\dots(x-a_n)^{q_n}}=\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\dots(y-b_n)^{q_n}}$$

defines a curve in which the angle made by the tangent with a fixed line is a linear combination of the angles made with that line by the rays drawn from its point of contact to a set of fixed points. Thus the discovery of any curve defined in this way is reduced to a question of quadrature; and on the other hand the integration of the differential equation is accomplished if the curves possessing this property are known. It is obvious, further, that the equation

$$e^{-i\alpha}\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\dots(x-a_n)^{q_n}}=e^{i\alpha}\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\dots(y-b_n)^{q_n}}$$

defines an oblique trajectory of the foregoing curves. I give a number of illustrations of this geometrical interpretation of such equations, treating in conclusion the curve defined by the equation

$$\sin x\,dx=\sin y\,dy,$$

which, when $\sin x$ is represented as an infinite product, falls under the above head. In connection with this case, I discuss the values of certain series which happen to be suggested by it, and which seemed, though not specially pertaining to the subject, to be of sufficient interest to warrant their consideration here.

I.—INTRODUCTORY.

If X, Y denote rectangular coordinates, the name of "circular coordinates" has been given to the quantities $X+iY$, $X-iY$, which we shall denote by x, y. Thus

$$x = X + iY = re^{i\vartheta}, \quad y = X - iY = re^{-i\vartheta}, \tag{1}$$

where r, ϑ are the ordinary polar coordinates of the point (X, Y); and hence

$$\frac{x}{y} = e^{2i\vartheta}. \tag{2}$$

If ϑ denote the angle made with $Y=0$ by the line joining the points 1 and 2, we have

$$e^{2i\vartheta} = \frac{x_2 - x_1}{y_2 - y_1}, \tag{3}$$

and, in particular, if ϑ be the angle made with $Y=0$ by the tangent to a curve at the point (x, y),

$$e^{2i\vartheta} = \frac{dx}{dy}. \tag{4}$$

We shall use the name *inclination* (or inclination with respect to the axis $Y=0$) for the angle ϑ; and when more than one line is concerned, the quantity $\Sigma\vartheta$* will be spoken of as the *inclination of the system of lines.* The function $e^{2i\vartheta}$ or $e^{2i\Sigma\vartheta}$ may be called the *clinant* (or inclination-function) of the line or system of lines. Thus the clinant of a system of lines is the product of the clinants of the separate lines.

It should be observed that if two sets of n lines each have equal clinants with respect to any axis, they have equal clinants with respect to any axis. To say that two systems of n lines each have equal clinants is the same as to say that they have the same orientation, or that the sum of the angles which the lines of one system make with an arbitrary axis is equal to the like sum for the other system. But relations between clinants other than that of equality—and even this relation when the systems do not consist of equal numbers of lines—depend upon the axis of reference.

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*It is understood throughout that $\Sigma\vartheta$ is determined only to modulus π. When two inclinations differ by a multiple of π, they are regarded as equal.

If the coordinates of a line be defined as the quantities u and v which occur in the equation

$$ux + vy + 1 = 0, \tag{5}$$

it is plain that the inclination of the line is given by the equation

$$e^{2i\vartheta} = -\frac{v}{u} \tag{6}$$

The clinant of the system of lines joining the origin to the points (x_1, y_1), (x_2, y_2), ... (x_n, y_n) is

$$e^{2i\Sigma\vartheta} = \frac{x_1 x_2 \ldots . x_n}{y_1 y_2 \ldots . y_n}, \tag{7}$$

and the clinant of the system of lines $u_1 x + v_1 y + 1 = 0, \ldots ., u_n x + v_n y + 1 = 0$ is

$$e^{2i\Sigma\vartheta} = (-)^n \frac{v_1 v_2 \ldots . v_n}{u_1 u_2 \ldots . u_n}, \tag{8}$$

A third variable will often be introduced for the sake of homogeneity; it will then be understood that $\frac{x}{z}$, $\frac{y}{z}$ must be used for the quantities above denoted by x and y; and $\frac{u}{w}$ and $\frac{v}{w}$ for the quantities denoted by u and v. The points (1, 0, 0), (0, 1, 0) are, of course, the circular points I, J.

II.—THE THEOREMS OF LAGUERRE AND HUMBERT.

The clinant of the asymptotes of the curve

$$A \equiv a_0 x^n + a_1 x^{n-1} y + \ldots . + a_{n-1} x y^{n-1} + a_n y^n + z A_1 = 0$$

is, by (7), $(-)^n \frac{a_n}{a_0}$. In order, then, that the asymptotes of another curve

$$B \equiv b_0 x^n + b_1 x^{n-1} y + \ldots . + b_{n-1} x y^{n-1} + b_n y^n + z B_1 = 0$$

have the same orientation, it is necessary and sufficient that

$$b_n : b_0 = a_n : a_0 .$$

But this is also the necessary and sufficient condition that in the pencil

$$\alpha A + \beta B = 0,$$

there be a curve in whose equation the group of terms free of z contains xy as a factor; i. e. a curve which passes through both the circular points. Hence the theorem:

In order that two curves of the n^{th} order have the same asymptotic orientation, it is necessary and sufficient that in the pencil determined by them there be included a circular curve.

If we consider the more general system

$$\alpha A + \beta B + \gamma C + \ldots . + \lambda L = 0,$$

where

$$C \equiv c_0 x^n + \ldots . + c_n y^n + zC_1, \ldots . , L \equiv l_0 x^n + \ldots . + l_n y^n + zL_1,$$

and α, β, ... λ are arbitrary parameters, the necessary and sufficient condition that the asymptotic orientation of the curves of this family be constant is obviously

$$\frac{a_n}{a_0} = \frac{b_n}{b_0} = \ldots . = \frac{l_n}{l_0}$$

But this is also the necessary and sufficient condition that any curve of the family which passes through I pass also through J. Hence the theorem:*

Given a family of curves of the n^{th} order whose equation involves any number of variable parameters; in order that the orientation of the asymptotes be the same for all these curves, it is necessary and sufficient that every curve of the family which passes through one of the circular points pass also through the other.

To find the clinant of the system of lines joining the origin to the intersections of two curves, we eliminate z between the two equations; the required clinant is the ratio of the coefficient of the highest power of y to the coefficient of the highest power of x in the resulting equation. Now these coefficients are unaffected by the terms containing xy in the equations of the given curves; hence the theorem:

In determining the orientation of the system of lines joining the origin to the intersections of a curve F_n with any other curve, F_n may be replaced by $F_n + xyF_{n-2}$, F_{n-2} being an arbitrary function of x, y, z of the degree $n - 2$. This may be otherwise stated as follows:

The orientation of the system of lines joining a given point O to the intersections of a curve F_n with any other curve, is unaltered if F_n be replaced by any curve of

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[*] Given by Humbert, this Journal, X, 260. Humbert, however, takes α, β, ... λ rational functions of one parameter; it is evident that this restriction would not interfere with the above proof.

the n^{th} order through the intersections of F_n with the circular rays of O; and, in particular, if F_n be replaced by *any system of n lines joining each of the n intersections of F_n with OI to one of the intersections of F_n with OJ.* If the curve is real, there is one and only one such system of n lines that is real.

The following special cases may be noticed:

1°. If O is a focus, and if (with Humbert) we designate as a *directrix* the line joining the points of contact of OI and OJ with the curve, the directrix counts twice among the system of n lines. In the case of a conic, therefore, the directrix counting twice entirely replaces the conic; whence the theorem that the inclination of the system of lines joining the focus of a conic to the intersections of the conic with any curve is twice the inclination of the system of lines joining the focus to the intersections of the directrix with the curve. If the cutting curve is a straight line, this becomes a familiar theorem; if the cutting curve is another conic having O for a focus, we see that the inclination of the system of lines joining O to the four intersections of the conics is four times the inclination of the line joining O to the intersection of the directrices.

2°. If O is on the curve, the tangent at O is one of the n replacing lines; if O is a k-ple point on the curve, the k tangents at O constitute k of the n replacing lines.

3°. If the curve passes l times through the circular points, the line at infinity counts l times among the n lines.

It is obvious from 2° and 3° that if we take as axis a line with respect to which the inclination of the tangents at O is 0,* the inclination of the lines joining O to the intersections of the curve F_n with any other curve C is equal to the inclination of the asymptotes of C counted l times, plus the inclination of the lines joining O to the intersection of C with a certain system of $n - k - l$ lines. In particular, if $k + l = n$, the inclination of the system $O(F_n, C)$ is simply l times that of the asymptotes of C. The following particular instances of this last case may be mentioned:

a). The sum of the angles made with a tangent to a circle by the lines joining its point of contact to the intersections of the circle with any curve is equal to the sum of the angles made with the tangent by the asymptotes of the curve.

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* Viz. the tangent itself if O is an ordinary point, a bisector of the angle between the tangents if O is a double point, etc.

b). Given a $\left\{\begin{matrix}\text{circular cubic}\\ \text{bicircular quartic}\end{matrix}\right.$ with a double point O, let OX bisect the angle between the tangents at O. Then the sum of the angles made with OX by the lines joining O to the intersections of the $\left\{\begin{matrix}\text{cubic}\\ \text{quartic}\end{matrix}\right.$ with any curve C is equal to $\left\{\begin{matrix}\text{the sum}\\ \text{twice the sum}\end{matrix}\right.$ of the angles made with OX by the asymptotes of the curve C. Thus, for example, the orientation of the lines joining the double point of a lemniscate to the four intersections of the lemniscate with any transversal depends only on the direction of the transversal; the sum of the angles made by the four lines with an axis of the lemniscate being twice the angle made with the axis by the transversal.

If we make no reference to any particular axis, the theorem of which the foregoing cases are illustrations has the form: If O be a k-ple point on a curve F_n which passes $n-k$ times through the circular points, the orientation of the system of lines joining O to the intersections of F_n with any curve C, depends only on the orientation of the asymptotes of C.

To find the clinant of the system of common tangents of two curves given in tangential coordinates, we should eliminate w between the equations of the curves; the clinant is the ratio of the coefficient of the highest power of u to that of the highest power of v in the resulting equation. But these coefficients are unaffected by the terms containing uv in the equations of the curves; hence, *in determining the orientation of the system of tangents common to a curve F_n and any other curve, F_n may be replaced by $F_n + uvF_{n-2}$*, F_{n-2} being an arbitrary function of u, v, w of the degree $n-2$.

On the other hand, to find a set of foci of a curve F_n not touching the line at infinity, we have to intersect each of the n lines given by the pair of equations

$$u=0, \quad F_n=0,$$

with one of the n lines given by the pair of equations

$$v=0, \quad F_n=0;$$

whence it is plain that if two curves have the same foci, their equations differ only in the terms involving uv; we have, therefore, this theorem of Laguerre's:

The orientation of the system of common tangents of two curves depends only on the position of the foci, and is therefore the same as that of the system of lines joining a set of foci of the one to a set of foci of the other. Obviously, this theorem may be regarded as dualistic to that of page 164 (end).

When one (or both) of the curves touches the line at infinity, this theorem requires some modification. If a curve touch the line at infinity at g points $u = \alpha_1 v$, ... $u = \alpha_g v$, *which do not include* I *or* J, its equation is of the form

$$(u - \alpha_1 v) \ldots . (u - \alpha_g v)\, w^{n-g} + \text{terms of lower degree in } w = 0;$$

and it is plain that the equation of any other curve having the same finite foci and the same contacts with the line at infinity will differ from this equation only in terms involving uv. Now, if we take any second curve $\Phi_{n'}$, touching the line at infinity at g' points,* the clinant of the system of $nn' - gg'$ common tangents of F_n and $\Phi_{n'}$ *other than the line at infinity* will be obtained as before through the elimination of w between the two equations, and will be unaffected by the terms containing uv in these equations. Hence *the orientation of the system of common tangents of two curves, exclusive of the line at infinity, is unaffected by replacing each curve by any other curve having the same finite foci and the same contacts with the line at infinity*; and, in particular, is the same as that of the system of lines (other than the line at infinity) joining the finite foci and the contacts with the line at infinity of the one curve with the finite foci and the contacts with the line at infinity of the other curve. For example, the three common tangents of two parabolas have the same orientation as the axes of the parabolas and the line joining their foci; and the four common tangents of a parabola and a circle have the same orientation as the axis of the parabola counted twice and the line joining the centre of the circle with the focus of the parabola, counted twice.

If, among the g contacts with the line at infinity the circular points are included, Laguerre's theorem requires further modification; we shall consider, however, only the extreme case of curves whose foci are all at infinity; the general equation of such curves is evidently

$$uvwF_{n-3}(u, v, w) + au^n + \ldots . + bv^n = 0.$$

Now the result of eliminating w between a given equation of this form and the equation of any curve of the class r not touching the line at infinity is evidently

$$a^r u^{nr} + \ldots . + b^r v^{nr} = 0,$$

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[*] These points I suppose to be different from the g points of contact belonging to F; if any of the latter coincide with any of the former set, complications are introduced which it does not seem worth while to discuss.

so that *the clinant of the system of common tangents, which is* a^r/b^r, *depends only on the value of* r, and is otherwise independent of the nature of the other curve. That is,

Given a curve F all of whose foci are at infinity, the orientation of the system of tangents common to it and any curve C not touching the line at infinity depends only on the class of C;[*] so that the orientation of the system of tangents to F through a point is independent of the position of the point, and the orientation of the system of tangents common to F and any curve of the r^{th} class not touching the line at infinity is the same as that of the system of tangets to F through r arbitrary points.

It is worth while to examine more explicitly the case of the tangents drawn *from a point* to a curve.

If the equation of a curve in tangential coordinates is

$$au^n + bv^n + cw^n + \ldots\ldots = 0,$$

the system of tangents through the origin is given by

$$w = 0, \quad au^n + \ldots\ldots + bv^n = 0,$$

and the clinant of this system of lines is a/b.

Observing that a change of origin is effected by replacing w by $w + \alpha u + \beta v$, we see at once that in order that the clinant of the system of tangents drawn to the curve from any point be constant, it is *necessary and sufficient* that the terms in w all contain the product uv; in other words, that the foci of the curve be all at infinity. This is a proof of Humbert's theorem independent of the proof given above.

The systems of tangents through I and J are given by the equations

$$v = 0, \; au^n + \ldots\ldots + cw^n = 0; \quad u = 0, \; bv^n + \ldots\ldots + cw^n = 0;$$

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[*] See Humbert, this Journal, X, 263. Humbert gives only the case where the second curve is a point; the more general theorem is, however, an obvious corollary from Humbert's. It should be added that Humbert makes an oversight in saying that curves which have all their foci at infinity touch the line at infinity $n-1$ times and pass through I and J; it is only necessary that they touch the line at infinity at I and J, and that the line at infinity count as *n tangents from I* (and likewise from J). Thus the line at infinity need not be more than a double tangent; and the curve may therefore have $n-2$ tangents parallel to a given line; while if the line at infinity were an $(n-1)$-fold tangent, the curve could have but one tangent parallel to a given line.

so that if $x_1, x_2, \ldots . x_n$ be the x's and $y_1, y_2, \ldots . y_n$ the y's of the foci, we have

$$x_1 x_2 \ldots . x_n = (-)^n \frac{a}{c}, \quad y_1 y_2 \ldots . y_n = (-)^n \frac{b}{c},$$

$$\frac{x_1 x_2 \ldots . x_n}{y_1 y_2 \ldots . y_n} = \frac{a}{b};$$

i. e. the clinant of the system of lines joining the origin to a group of foci is a/b, which is the same as that of the system of tangents from the origin; hence the system of tangents drawn from any point to a curve and the system of lines joining that point to a group of foci, have the same orientation: a particular case of Laguerre's theorem (page 166, end).

Denoting the lengths of the focal radii of the origin—i. e. the lines joining the origin to a set of foci—by $r_1, r_2, \ldots . r_n$, we have, since $r_k^2 = x_k y_k$,

$$(r_1 r_2 \ldots . r_n)^2 = x_1 x_2 \ldots . x_n . y_1 y_2 \ldots . y_n = \frac{ab}{c^2}.$$

If, then, two curves have the coefficients a, b in the same ratio, the orientation of the system of lines joining the origin to a group of foci is the same for the two curves; and if they have the coefficients a, b, c in the same ratio, the product of the focal radii of the origin is also the same for both curves. But the equality of the ratio of a to b for two curves is evidently the condition that in the tangential pencil determined by them there be one which has the origin for a focus; and the equality of the ratios of a, b, c for two curves is the condition that in the tangential pencil determined by them there be one which touches the line at infinity and has the origin for a focus. Hence we have the theorem:

If, from a given focus of a curve belonging to a tangential pencil, lines be drawn to the foci of any curve of the pencil, the orientation of this system of lines is constant; and if the curve to which the given focus belongs touches the line at infinity, the product of the lengths of the lines drawn from it is also constant.* For example, in a system of conics touching four lines, the bisector of the angle formed by the lines joining a focus of a given conic of the system to the two foci of any other conic of the system is a fixed line; and if the given conic is the parabola that belongs to the system, the product of the distances from its focus to the two foci of any other conic of the system is also constant.

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*I. e., any complete set of foci, for instance the real foci.

And it is evident, conversely, that

If a point O be such that the systems of lines joining it to the foci of two curves have the same orientation, O is a focus of a curve belonging to the tangential pencil determined by the two curves; and if the product of the lengths of the lines be also the same for the two systems, the curve of which O is a focus touches the line at infinity.

The property contained in the first clause of this and the preceding theorem was given by Humbert (this Journal, X, 262); he points out that it defines by a simple geometrical character the locus of the foci of a tangential pencil.

The equation of this locus is very easily obtained. In fact, the foci of the curve $F(u, v, w) = 0$ are evidently given by the equations $F(z, 0, -x)$, $F(z, 0, -y) = 0$. Hence any focus of a curve belonging to the pencil

$$F(u, v, w) + \lambda\Phi(u, v, w) = 0$$

satisfies the equations

$$F(z, 0, -x) + \lambda\Phi(z, 0, -x) = 0, \quad F(0, z, -y) + \lambda\Phi(0, z, -y) = 0;$$

hence the locus of the foci is

$$F(z, 0, -x)\,\Phi(0, z, -y) - F(0, z, -y)\,\Phi(z, 0, -x) = 0.$$

It is evident that this equation of the $(2n)^{\text{th}}$ degree contains the factor z, and that after striking out this factor, every term is of at least the $(n-1)^{\text{th}}$ degree in x and z and of at least the $(n-1)^{\text{th}}$ degree in y and z.

The tangents to this locus at the circular points are given by the aggregate of terms of lowest degree in (x, z) and (y, z) respectively, that is, by the equations

$$\frac{1}{z}\{F(z, 0, -x)\,\Phi(0, 0, -1) - \Phi(z, 0, -x)\,F(0, 0, -1)\} = 0,$$

$$\frac{1}{z}\{F(0, z, -y)\,\Phi(0, 0, -1) - \Phi(0, z, -y)\,F(0, 0, -1)\} = 0.$$

The intersections of these two pencils of lines are the singular foci of the locus; and these intersections obviously satisfy the equation

$$F(z, 0, -x)\,\Phi(0, z, -y) - F(0, z, -y)\,\Phi(z, 0, -x) = 0,$$

that is, they lie on the curve itself. Hence

The locus of the foci of the curves belonging to the tangential pencil determined by two curves of the n^{th} class is (apart from the line at infinity) *a curve of the*

$(2n-1)^{th}$ *order which has the circular points for* $(n-1)$*-ple points and which passes through its own singular foci.**

If the curves F and Φ both touch the line at infinity, or if one of them touch it more than once, the locus degenerates. For example, if the curves F and Φ each have the line at infinity as a simple tangent,

$$F(z,\ 0,\ -x),\quad F(0,\ z,\ -y),\quad \Phi(z,\ 0,\ -x),\quad \Phi(0,\ z,\ -y)$$

each contain z once as a factor; and the locus

$$F(z,\ 0,\ -x)\Phi(0,\ z,\ -y)-F(0,\ z,\ -y)\Phi(z,\ 0,\ -x)=0$$

includes the line at infinity counted twice, and *a curve of the* $(2n-2)^{th}$ *order which has the circular points for* $(n-1)$*-ple points.* Thus the locus of the foci of the parabolas which touch three given lines is a circle, and the locus of the foci of the curves of the third class which touch the line at infinity and seven other given lines is a bicircular quartic.

Given a number of points $(x_1,\ y_1),\ \ldots .\ (x_n,\ y_n)$, let the point defined by the equations

$$\frac{1}{x}=\frac{1}{n}\left(\frac{1}{x_1}+\frac{1}{x_2}+\ldots .+\frac{1}{x_n}\right),\quad \frac{1}{y}=\frac{1}{n}\left(\frac{1}{y_1}+\frac{1}{y_2}+\ldots .+\frac{1}{y_n}\right)$$

be called the harmonic centre of the system of points $1,\ 2,\ \ldots .\ n$ with respect to the origin. It is plain that if the points are real (so that x_i and y_i are conjugate imaginaries), the harmonic centre C may be constructed by laying off on $O1,\ O2,\ \ldots .\ On$, distances equal to the reciprocals of $O1,\ O2,\ \ldots .\ On$, and taking for OC the reciprocal of the n^{th} part of the resultant of the system of forces represented by $O1,\ O2,\ \ldots .\ On$.

The direction of the harmonic centre depends only on the ratio of $\sum\frac{1}{x_1}$ to $\sum\frac{1}{y_1}$; its distance only on the product of $\sum\frac{1}{x_1}$ by $\sum\frac{1}{y_1}$.

The harmonic centre $(x,\ y)$ of a set of foci of the curve

$$au^n+a_1u^{n-1}w+\ldots .+bv^n+b_1v^{n-1}w+\ldots .=0,$$

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*Humbert (l. c., p. 277) gives this theorem for the particular case of a tangential pencil of conics; he deduces it from a construction of the locus by points, which shows the locus to be a circular cubic on which I and J are corresponding points.

is given by

$$\frac{n}{x}=\sum\frac{1}{x_1}=\frac{a_1}{a},\quad \frac{n}{y}=\sum\frac{1}{y_1}=\frac{b_1}{b};$$

and for any curve of the tangential pencil determined by the above curve and the curve

$$\alpha u^n+\alpha_1u^{n-1}w+\ldots.+\beta v^n+\beta_1v^{n-1}w+\ldots.=0$$

the harmonic centre of the foci is given by

$$\frac{n}{x}=\frac{a_1+\lambda\alpha_1}{a+\lambda\alpha},\quad \frac{n}{y}=\frac{b_1+\lambda\beta_1}{b+\lambda\beta},$$

whence, eliminating λ, the locus of the harmonic centres of foci for the curves of the pencil is

$$(a_1\beta_1-\alpha_1b_1)\,xy-n\,(a_1\beta-\alpha_1b)\,x-n\,(a\beta_1-\alpha b_1)\,y+n^2\,(a\beta-\alpha b)=0,$$

a circle. Hence the theorem (given by Humbert without demonstration, l. c., p. 281):

The harmonic centre, with respect to a point of the plane, of the real foci of each of the curves of a tangential pencil, describes a circle.

Concerning this locus of harmonic centres, the following points are obvious:

1°. The locus passes through the origin if $a:b=\alpha:\beta$; i. e. if the origin is a focus of a curve of the pencil.

2°. The locus becomes a straight line if $a_1:b_1=\alpha_1:\beta_1$; i. e. if for one of the curves the harmonic centre is at an infinite distance from the origin.

3°. The harmonic centre is a fixed point for all the curves of the pencil if $a:a_1:b:b_1=\alpha:\alpha_1:\beta:\beta_1$; i. e. if the origin is a double or multiple focus of some curve of the pencil. Finally, observing that the equation of the locus of the foci of the curves of the pencil (see p. 170) is, apart from the line at infinity,

$$(a\beta-\alpha b)\,z^{2n-1}-\{(a_1\beta-\alpha_1b)\,x+(a\beta_1-\alpha b_1)\,y\}\,z^{2n-2}+\ldots.=0,$$

we see that when $a:b=\alpha:\beta$ this curve not only passes through the origin but it has at the origin the same tangent as the circle of harmonic centres; whence this theorem of Humbert's (l. c., p. 276):

Let F be the locus of the foci of the curves of class n belonging to a given tangential pencil: the harmonic centre of the n real foci of any one of these curves with respect to a point arbitrarily chosen upon F remains on a circle touching the curve F at this point.

III.—ON THE DIFFERENTIAL EQUATION

$$\frac{dx}{(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots\ldots(x-a_n)^{q_n}}=\frac{dy}{(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots\ldots(y-b_n)^{q_n}}.$$

General Remarks and Miscellaneous Examples.

The differential equation

$$\frac{dx}{\phi(x)}=\frac{dy}{\psi(y)}, \tag{1}$$

where

$$\phi(x)=(x-a_1)^{q_1}(x-a_2)^{q_2}\ldots(x-a_n)^{q_n},\quad \psi(y)=(y-b_1)^{q_1}(y-b_2)^{q_2}\ldots(y-b_n)^{q_n},$$

admits of a simple and interesting geometrical interpretation if x, y are regarded as circular coordinates. Namely, writing the equation

$$\frac{dx}{dy}=\frac{\phi(x)}{\psi(y)},$$

we see that it defines a curve having the property

$$\vartheta_T=q_1\vartheta_1+q_2\vartheta_2+\ldots\ldots q_n\vartheta_n, \tag{2}$$

ϑ_T being the angle made with the axis of X by the tangent at any point P, and ϑ_k the angle made with the axis of X by the line joining P to the point (a_k, b_k).

Obviously, the curves belonging to the equation

$$\frac{dx}{\phi(x)}+\frac{dy}{\psi(y)}=0 \tag{3}$$

have the property (ϑ_N denoting the inclination of the normal)

$$\vartheta_N=q_1\vartheta_1+q_2\vartheta_2+\ldots\ldots+q_n\vartheta_n, \tag{4}$$

and are the orthogonal trajectories of the former set of curves; and, more generally, the oblique trajectories of the curves

$$\frac{dx}{\phi(x)}=\frac{dy}{\psi(y)},$$

the angle of intersection being α, are given by

$$e^{-i\alpha}\frac{dx}{\phi(x)}=e^{i\alpha}\frac{dy}{\psi(y)}. \tag{5}$$

The curves defined by the last equation have the property

$$\vartheta_T=\alpha+q_1\vartheta_1+q_2\vartheta_2+\ldots\ldots+q_n\vartheta_n. \tag{6}$$

When the q's are all integers, positive or negative, ϕ and ψ are rational and the equation is integrable; hence we can always obtain the equation of a curve defined by the property that the angle made by its tangent with a fixed line is a sum of integer multiples of the angles made with that line by the rays from the point of contact to n fixed points, which we may call direction-centres. If the q's, besides being integers, are such that

$$q_1 + q_2 + \dots + q_n = 1,$$

the equation giving the inclination of the tangent will not be affected by changing the directions of the axes of reference; and the property of the tangent of the curve given by equation (1) may be stated thus: the system of lines consisting of the tangent and the rays from its point of contact to the direction-centres corresponding to negative exponents, has the same orientation as the system of rays from the point of contact to the direction-centres corresponding to positive exponents; each direction-centre being counted a number of times equal to the absolute value of the corresponding exponent.

Let us consider some examples.

1°. $\dfrac{dx}{x} - \dfrac{dy}{y} = 0$. This equation defines a curve in which the *tangent coincides with the radius vector.* Its solution is $y = cx$ or $Y = CX$, a straight line through the origin.

2°. $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$; orthogonal trajectory of preceding; *normal coincides with radius vector.* Solution: $xy = c$ or $X^2 + Y^2 = C$, a circle with its centre at the origin.

3°. $\dfrac{xdx}{x^2 - a^2} - \dfrac{ydy}{y^2 - a^2} = 0$; *bisector of angle formed by tangent and radius vector coincides with bisector of angle formed by rays to the two points* $X = \pm a$, $Y = 0$. Solution: $x^2 - a^2 = c(y^2 - a^2)$, or $X^2 - Y^2 - a^2 = CXY$, an equilateral hyperbola passing through the two points and having its centre at the origin.

4°. $\dfrac{xdx}{x^2 - a^2} + \dfrac{ydy}{y^2 - a^2} = 0$; orthogonal trajectory of preceding; *bisector of angle formed by normal and radius vector coincides with bisector of angle formed by rays to the two points* $X = \pm a$, $Y = 0$. Solution: $(x^2 - a^2)(y^2 - a^2) = c$, or $(X^2 + Y^2)^2 - 2a^2(X^2 - Y^2) = C$, a Cassinian with the two points for foci.

5°. $\frac{x^{n-1}dx}{x^n - a^n} - \frac{y^{n-1}dy}{y^n - a^n} = 0$; *orientation of tangent and* $(n - 1)$ *times radius vector is equal to orientation of rays to a system of n points (beginning with the point* $X = a$, $Y = 0$) *uniformly distributed on a circle whose centre is the origin.* Solution: $x^n - a^n = c(y^n - a^n)$; or, in polar coordinates, $r^n \cos n\vartheta - a^n = Cr^n \sin n\vartheta$.

6°. $\frac{x^{n-1}dx}{x^n - a^n} + \frac{y^{n-1}dy}{y^n - a^n} = 0$; orthogonal trajectory of preceding; *orientation of normal and* $(n - 1)$ *times radius vector is equal to orientation of rays to the system of n points defined in* 5°. Solution: $(x^n - a^n)(y^n - a^n) = c$; or, in polar coordinates, $r^{2n} - 2a^n r^n \cos n\vartheta = C$. The curves $r^n = 2a^n \cos n\vartheta$, $r^n \cos n\vartheta = a^n$, are particular solutions of 6° and 5° respectively; when $n = 2$, these are a lemniscate and an equilateral hyperbola with its vertices at the foci of the lemniscate.

Of course, whenever $q_1 + q_2 + \ldots + q_n = 1$, whether the q's be integers or not, the property of the tangent is independent of the direction of reference; e. g., the equation

$$\frac{dx}{\sqrt{a + 2bx + cx^2}} + \frac{dy}{\sqrt{a + 2by + cy^2}} = 0$$

gives a curve whose normal bisects the angle between the rays drawn to two fixed points; and, more generally, the equation

$$\frac{x^{n-1}dx}{\sqrt{P_{2n}(x)}} + \frac{y^{n-1}dy}{\sqrt{P_{2n}(y)}} = 0,$$

$P_{2n}(x)$ being a polynomial of the $(2n)^{\text{th}}$ degree in x, gives a curve such that the sum of the angles made with *any* line by the normal and by the radius vector counted $(n - 1)$ times is half the sum of the angles made with that line by the rays to $2n$ fixed points.

If the sum of the q's is not equal to 1, the equation connecting the inclination of the tangent with the inclinations of the rays to the direction-centres will be modified if the direction of reference is altered; the modification consists, however, merely in adding a constant to the value of ϑ_T.

In the examples that follow, when the word *inclination* is used, it is understood to have reference to the axis of X.

7°. $\frac{dx}{x^{n+1}} - \frac{dy}{y^{n+1}} = 0$; *inclination of tangent* $= (n + 1)$ *times inclination of radius vector.* Solution: $x^{-n} - y^{-n} = c$, or $r^n = C^n \sin n\vartheta$.

8°. $\frac{dx}{x^{n+1}} + \frac{dy}{y^{n+1}} = 0$;* *inclination of normal* $= n + 1$ *times inclination of radius vector.* Solution: $r^n = C^n \cos n\vartheta$. Of course, in 7° and 8°, n need not be an integer.

We may remark that a particular solution of 8° coincides with a particular solution of 6°, viz. the solution $r^n = 2a^n \cos n\vartheta$; in this curve, then, we have, if n is a positive integer,

$$\vartheta_N = (n+1)\,\vartheta_0 \text{ and } \vartheta_N + (n-1)\,\vartheta_0 = \vartheta_1 + \vartheta_2 + \dots + \vartheta_n,$$

whence also

$$2n\vartheta_0 = \vartheta_1 + \vartheta_2 + \dots + \vartheta_n;$$

so that, for instance, in the lemniscate the sum of the angles made by the focal radii with the axis is 4 times the angle made by the radius vector with the axis.

We may here notice a point concerning the Cassinian generally. The equation being written

$$(X^2 + Y^2)^2 - 2a^2(X^2 - Y^2) = b^4 - a^4,$$

the two foci, 1 and 2, by means of which the curve is usually defined, are on the axis of X at the distance a on either side of the origin; and it is evident on inspection of the equation written in the form

$$x^2y^2 - a^2(x^2 + y^2)\,z^2 + (a^4 - b^4)\,z^4 = 0$$

that these two foci are at the same time the two singular foci† and two of the ordinary foci, the remaining two ordinary foci, 3 and 4, being on the axis of Y at the distance $\sqrt{b^4 - a^4}/a$ on either side of the origin if $b > a$, and on the axis of X at the distance $\sqrt{a^4 - b^4}/a$ on either side of the origin if $b < a$; of course, if $b = a$, i. e. if the Cassinian is a lemniscate, 3 and 4 coincide with the origin. Now by a known property belonging to bicircular quartics in general,

$$2\vartheta_N = \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4,$$

and by example 6°

$$\vartheta_N + \vartheta_0 = \vartheta_1 + \vartheta_2.$$

Hence

$$\vartheta_N - \vartheta_0 = \vartheta_3 + \vartheta_4,$$

and

$$2\vartheta_0 = \vartheta_1 + \vartheta_2 - \vartheta_3 - \vartheta_4.$$

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* This is, of course, the orthogonal trajectory of the preceding; but it is also plain that the curves corresponding to this equation are simply those corresponding to 7° turned through an angle $\frac{\pi}{2n}$.

† I. e., foci obtained by intersecting tangents *at* I and J. It is because I and J are inflexions that these foci play the double part in the Cassinian.

In the particular case of the lemniscate, $\vartheta_3 = \vartheta_4 = \vartheta_0$, and this equation becomes $4\vartheta_0 = \vartheta_1 + \vartheta_2$, as otherwise found above.

If $\int \frac{dx}{f(x)} = F^{-1}(x)$, and if $F^{-1}(X - iY)$ may be taken the conjugate of $F^{-1}(X + iY)$, then writing

$$F^{-1}(X + iY) = u + iv,$$

the solution of

$$e^{-i\alpha} \frac{dx}{f(x)} + e^{i\alpha} \frac{dy}{f(y)} = 0$$

may be written

$$u \cos\alpha + v \sin\alpha = C.$$

If, further, we write

$$F(u + iv) = F_1(u, v) + iF_2(u, v),$$

the curve is given in rectangular coordinates by the equations

$$X = F_1(u, v), \quad Y = F_2(u, v),$$

the parameters being connected by the relation

$$u \cos\alpha + v \sin\alpha = C.$$

The next two examples may serve as illustrations of the foregoing.

9°. $e^{-i\alpha} \frac{dx}{\sqrt{k^2 - x^2}} + e^{i\alpha} \frac{dx}{\sqrt{k^2 - y^2}} = 0$; *normal makes an angle* α *with bisector of rays to points* $X = \pm k$, $Y = 0$. Here we may take

$$F(u + iv) = k \cos(u + iv) = k(\cos u \cosh v - i \sin u \sinh v);$$

hence the curves are given by

$$\begin{cases} X = k \cos u \cosh v, \quad Y = -k \sin u \sinh v, \\ \qquad u \cos\alpha + v \sin\alpha = C. \end{cases}$$

If $\alpha = 0$, $u = C$; if $\alpha = \frac{\pi}{2}$, $v = C$; in these cases the equation of the curves is evidently

$$\frac{X^2}{(k \cos C)^2} - \frac{Y^2}{(k \sin C)^2} = 1, \quad \frac{X^2}{(k \cosh C)^2} + \frac{Y^2}{(k \sinh C)^2} = 1,$$

either of which represents a system of confocal ellipses and hyperbolas; either equation is converted into the other by the substitution of iC for C.

It is easy to eliminate u and v in the general case; viz. we have

$$2\frac{X^2+Y^2}{k^2} = 2(\cos^2 u \cosh^2 v + \sin^2 u \sinh^2 v) = \cos 2u + \cosh 2v,$$

$$2\frac{X^2-Y^2}{k^2} = 2(\cos^2 u \cosh^2 v - \sin^2 u \sinh^2 v) = 1 + \cos 2u \cosh 2v,$$

so that $\cos 2u$, $\cosh 2v$ are the roots of the equation

$$k^2\lambda^2 - 2(X^2+Y^2)\lambda + 2(X^2-Y^2) - k^2 = 0.$$

The equation of the curve is therefore

$$\cos\alpha \cos^{-1}\lambda_1 + \sin\alpha \cosh^{-1}\lambda_2 = C,$$

λ_1 and λ_2 being the roots of the preceding equation; or, more explicitly,

$$\cos\alpha \cos^{-1}\frac{X^2+Y^2-\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2}$$
$$+\sin\alpha \cosh^{-1}\frac{X^2+Y^2+\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2} = C.$$

When $\alpha = 0$ or $\frac{\pi}{2}$, one value of λ is constant, and the equation of the curves may be obtained by putting $\lambda = c$ in the quadratic equation for λ; when we get

$$k^2c^2 - 2(X^2+Y^2)c + 2(X^2-Y^2) - k^2 = 0,$$
$$2X^2(c-1) + 2Y^2(c+1) = k^2(c^2-1),$$

or

$$\frac{X^2}{\frac{1}{2}k^2(c+1)} + \frac{Y^2}{\frac{1}{2}k^2(c-1)} = 1,$$

a system of conics with their foci at $X = \pm k$, $Y = 0$.

10°. $e^{-i\alpha}\dfrac{dx}{x^2-k^2} + e^{i\alpha}\dfrac{dy}{y^2-k^2} = 0$; *normal makes an angle* α *with line whose inclination = inclination of the pair of rays to the points* $X = \pm k$, $Y = 0$. Here we may take

$$F(u+iv) = k\tanh(u+iv) = k\frac{\sinh u \cos v + i\cosh u \sin v}{\cosh u \cos v + i \sinh u \sin v}$$
$$= k\frac{\sinh u \cosh u + i \sin v \cos v}{\cosh^2 u \cos^2 v + \sinh^2 u \sin^2 v}$$
$$= k\frac{\sinh 2u + i \sin 2v}{\cosh 2u + \cos 2v}$$

Hence the curves are given by

$$\begin{cases} X = k\dfrac{\sinh 2u}{\cosh 2u + \cos 2v}, \quad Y = k\dfrac{\sin 2v}{\cosh 2u + \cos 2v}, \\ u\cos\alpha + v\sin\alpha = C. \end{cases}$$

The parameters u and v are easily eliminated as follows:

$$\frac{X^2+Y^2}{k^2} = \frac{\sinh^2 2u + \sin^2 2v}{(\cosh 2u + \cos 2v)^2} = \frac{\cosh^2 2u - \cos^2 2v}{(\cosh 2u + \cos 2v)^2} = \frac{\cosh 2u - \cos 2v}{\cosh 2u + \cos 2v}.$$

Hence

$$\frac{\cosh 2u}{\cosh 2u + \cos 2v} = \frac{X^2+Y^2+k^2}{2k^2}, \quad \frac{\cos 2v}{\cosh 2u + \cos 2v} = \frac{X^2+Y^2-k^2}{-2k^2},$$

so that, since the original equations give

$$\frac{\sinh 2u}{\cosh 2u + \cos 2v} = \frac{X}{k}, \quad \frac{\sin 2v}{\cosh 2u + \cos v} = \frac{Y}{k},$$

we have

$$\tanh 2u = \frac{2kX}{X^2+Y^2+k^2}, \quad \tan 2v = \frac{-2kY}{X^2+Y^2-k^2};$$

substituting these values in the equation $u\cos\alpha + v\sin\alpha = C$, we have the equation of the curves:

$$\cos\alpha \tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha \tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C.$$

If $\alpha = \dfrac{\pi}{2}$, this becomes

$$X^2+Y^2-k^2 = cY,$$

a system of circles through the points $Y = 0$, $X = \pm k$. If $\alpha = \dfrac{\pi}{2}$, it becomes

$$X^2+Y^2+k^2 = cX,$$

the orthogonal system of circles, viz. those through $X = 0$, $Y = \pm ik$, the anti-points of the preceding pair of points.

Thus the oblique trajectory of a system of circles through two fixed points, like the oblique trajectory of a system of confocal ellipses, is a transcendental curve; the equations of the trajectories being, as found above, respectively

$$\cos\alpha \tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha \tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C,$$

$$\cos\alpha \cos^{-1}\frac{X^2+Y^2-\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2}$$

$$+\sin\alpha \cosh^{-1}\frac{X^2+Y^2+\sqrt{(X^2+Y^2)^2-2k^2(X^2-Y^2)+k^4}}{k^2} = C.$$

But the former reduces to an algebraic curve in the particular case when the two common-points of the circles coincide, while the latter does not do so when the two foci coincide. To get the limiting form of the first equation when $k = 0$, we may write it

$$\cos\alpha \,.\, \frac{1}{k}\tanh^{-1}\frac{2kX}{X^2+Y^2+k^2} - \sin\alpha \,.\, \frac{1}{k}\tan^{-1}\frac{2kY}{X^2+Y^2-k^2} = C',$$

which, when $k = 0$, becomes

$$\frac{2X\cos\alpha}{X^2+Y^2} - \frac{2Y\sin\alpha}{X^2+Y^2} = C',$$

or

$$C'(X^2+Y^2) = 2(X\cos\alpha - Y\sin\alpha),$$

a system of circles touching the line $X\cos\alpha - Y\sin\alpha = 0$ at the origin.

The second equation may be written, if we put, for brevity,

$$X^2+Y^2 = p, \quad X^2-Y^2 = q,$$

$$\cos\alpha\cos^{-1}\frac{p-\sqrt{p^2-2k^2q+k^4}}{k^2} + \sin\alpha\left(\cosh^{-1}\frac{p+\sqrt{p^2-2k^2q+k^4}}{k^2} - \cosh^{-1}\frac{2c}{k^2}\right) = 0;$$

when k is infinitesimal, this may be written

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\left(\cosh^{-1}\frac{2p}{k^2} - \cosh^{-1}\frac{2c}{k^2}\right) = 0,$$

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\cosh^{-1}\left(\frac{2p}{k^2}\cdot\frac{2c}{k^2} - \sqrt{\frac{4p^2}{k^4}-1}\sqrt{\frac{4c^2}{k^4}-1}\right) = 0,$$

which gives, in the limit,

$$\cos\alpha\cos^{-1}\frac{q}{p} + \sin\alpha\cosh^{-1}\left(\frac{p}{2c}+\frac{c}{2p}\right) = 0.$$

Now $\cosh^{-1}\left(\frac{p}{2c}+\frac{c}{2p}\right) = \log\frac{p}{c} = \log\frac{X^2+Y^2}{c} = \log\frac{r^2}{c} = 2\log\frac{r}{c'}$; and $\cos^{-1}\frac{q}{p} = \cos^{-1}\frac{X^2-Y^2}{X^2+Y^2} = \cos^{-1}(\cos 2\vartheta) = 2\vartheta$; hence the above equation becomes (in polar coordinates)

$$r = c'e^{-\vartheta\cot\alpha}$$

a system of similar logarithmic spirals. Of course, the trajectories for the case when the two direction-centres coincide are in both instances obvious otherwise.

11°. If a curve be defined by the property that the inclination of the tangent is equal to half the inclination of the system of rays drawn from its point of contact to four arbitrary fixed points, its differential equation is

$$\frac{dx}{\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)}}=\frac{dy}{\sqrt{(y-b_1)(y-b_2)(y-b_3)(y-b_4)}},$$

the four points being $(a_1, b_1) \ldots . (a_4, b_4)$.

Denoting the function inverse to the integral of the first member by ϕ, and the function inverse to the integral of the second member by ψ, the solution of this equation is

$$\phi^{-1}(x)-\psi^{-1}(y)=C.$$

If the four points are real, $b_1 \ldots . b_4$ are the conjugates of $a_1 \ldots . a_4$ respectively; and consequently if we write $\phi^{-1}(X+iY)=u_1+iu_2$, we may put $\psi^{-1}(X-iY)=u_1-iu_2$, so that the foregoing equation becomes

$$u_2=c.$$

Writing, then,

$$\phi(u_1+iu_2)=\phi_1(u_1, u_2)+i\phi_2(u_1, u_2),$$

the curve is given in rectangular coordinates by the equations

$$\begin{cases} X=\phi_1(u_1, c), \\ Y=\phi_2(u_1, c). \end{cases}$$

The solution of the problem thus depends solely on the separation of $\phi(u_1+iu_2)$ into its real and pure imaginary components: ϕ being the function inverse to the general elliptic integral of the first kind, $\int dx/\sqrt{R(x)}$, where $R(x)$ is a quartic whose coefficients are in general complex.

It would have been no more difficult to consider the oblique trajectories of these curves; but in point of fact these furnish nothing new, since by rotation of the axes of reference they are reducible to the case just considered.*

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*This is not the same as to say that *with a given set of four points and a given axis of reference*, the oblique trajectories are of the same nature as the original curves: which is not true. But with a proper choice of the axis of reference, any proposed case of the oblique trajectory is reduced to *another* case of the original class of curves.

Nor would it be more difficult to consider the case when the four arbitrary points are not real. The proposed curves would in this case be given by

$$\begin{cases} X = \phi_1(u_1, u_2) = \quad \psi_1(v_1, v_2), \\ Y = \phi_2(u_1, u_2) = -\psi_2(v_1, v_2), \\ \quad u_1 - v_1 + i(u_2 - v_2) = 0, \end{cases}$$

ψ_1 and ψ_2 being of course defined by the equation

$$\psi(v_1 + iv_2) = \psi_1(v_1, v_2) + i\psi_2(v_1, v_2)$$

If the points $(a_1, b_1) \ldots . (a_4, b_4)$ are concyclic and are such that an "axis" of the four points is parallel to the axis of X, it is known* that the required curve is a bicircular quartic of which the four points are a set of ordinary foci. Hence in this case the problem of solving the differential equation may be reduced to that of finding the equation of a system of confocal bicircular quartics whose foci are any four concyclic points. It is evident that the condition that the four given points be concyclic is equivalent to the requirement that the quartic in x be linearly transformable into the quartic in y; so that the case to which this geometrical method of solving the differential equation is applicable is that in which the two quartics are homographic and satisfy a certain additional condition. It is obvious, moreover, that *all* the conditions of the case are expressed by the equations

$$\frac{(a_1 - a_2)(a_3 - a_4)}{(b_1 - b_2)(b_3 - b_4)} = \frac{(a_1 - a_3)(a_2 - a_4)}{(b_1 - b_3)(b_2 - b_4)} = \frac{(a_1 - a_4)(a_2 - a_3)}{(b_1 - b_4)(b_2 - b_3)} = 1,$$

viz. the equation of these fractions to 1 expresses that the inclination of the pair of lines 12, 34 is 0, and likewise for the pairs 13, 24 and 14, 23; and it follows incidentally that the four points are concyclic. This follows either by a well-known theorem of geometry, or from the fact that the equality of any two of the fractions shows the quartics to be homographic and the points consequently to be concyclic.

On the Equation $\sin x\,dx = \sin y\,dy$.

It is somewhat interesting to consider a case in which $f(x)$ is a convergent infinite product, as e. g. in the equation

$$\sin x\,dx = \sin y\,dy.$$

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*See "Note on the Double Periodicity of the Elliptic Functions," this Journal, XI, 285.

This equation defines a curve in which the angle made by the tangent with the axis of X is the negative of the sum of the angles made with the axis of X by the rays drawn to the points (on that axis)

$$X = 0,\ \pm\pi,\ \pm 2\pi,\ \pm 3\pi,\ \dots\,;$$

or, as we may write it,

$$-\vartheta_T = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \dots\text{*}$$

The curves thus defined have for their equation

$$\cos x - \cos y = c, \quad \text{i. e.} \quad \sin X \sinh Y = C.$$

If we compare the value of the inclination of the tangent as derived from this last equation with that which arises from the definition of the curve, we obtain an interesting identity, viz. we get in these ways respectively

$$\vartheta_T = \tan^{-1}\frac{dY}{dX} = -\tan^{-1}\frac{\cos X \sinh Y}{\sin X \cosh Y} = -\tan^{-1}\frac{\tanh Y}{\tan X},$$

$$\begin{aligned}-\vartheta_T = {} & \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{Y}{X-\pi} + \tan^{-1}\frac{Y}{X+\pi} \\ & + \tan^{-1}\frac{Y}{X-2\pi} + \tan^{-1}\frac{Y}{X+2\pi} + \dots \\ = {} & \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2-Y^2-\pi^2} + \tan^{-1}\frac{2XY}{X^2-Y^2-(2\pi)^2} + \dots\,;\end{aligned}$$

whence the identity

$$\begin{aligned}\tan^{-1}\frac{\tanh Y}{\tan X} \equiv {} & \tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2-Y^2-\pi^2} \\ & + \tan^{-1}\frac{2XY}{X^2-Y^2-(2\pi)^2} + \dots \pmod{\pi}\text{†} \qquad \text{(A)}\end{aligned}$$

or

$$\tanh Y = \tan X \tan\left\{\tan^{-1}\frac{Y}{X} + \tan^{-1}\frac{2XY}{X^2-Y^2-\pi^2} + \tan^{-1}\frac{2XY}{X^2-Y^2-(2\pi)^2} + \dots\right\}.$$

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* From the nature of the case (viz. because the equation $dx/dy = \sin y/\sin x$ determines a definite direction of the tangent, and because $\sin y/\sin x$ is truly the limit of $\frac{y}{x}\cdot\frac{y-\pi}{x-\pi}\cdot\frac{y+\pi}{x+\pi}\cdot\frac{y-2\pi}{x-2\pi}\cdot\frac{y+2\pi}{x+2\pi}\dots$, the factors being taken in this order) the series to the right converges to a definite limit, *modulo* π; and it is plainly best to take each pair of terms, $\vartheta_r + \vartheta_{-r}$, as near 0 as possible; then $\lim(\vartheta_n + \vartheta_{-n}) = 0$, obviously.

† It is needless to say that this identity is easily obtained without the intervention of geometry; but it is naturally suggested by the construction of the tangent to the curve we are considering.

In particular, when $X = Y$, we have

$$\tan^{-1}\frac{\tanh X}{\tan X} \equiv \frac{\pi}{4} - \tan^{-1}\frac{2X^2}{\pi^2} - \tan^{-1}\frac{2X^2}{(2\pi)^2} - \tan^{-1}\frac{2X^2}{(3\pi)^2} - \ldots (\mathrm{mod}\,\pi) \quad \text{(B)}$$

and when $X = \frac{\pi}{2}$,

$$\tan^{-1}\frac{1}{2.1^2} + \tan^{-1}\frac{1}{2.2^2} + \tan^{-1}\frac{1}{2.3^2} + \tan^{-1}\frac{1}{2.4^2} + \ldots . = \frac{\pi}{4} + \lambda\pi,$$

λ being an undetermined integer. If we take each of the inverse tangents at its least positive value $\left(\text{i. e. between 0 and } \frac{1}{2}\pi\right)$, it is easy to see that $\lambda = 0$; for the series on the left is less than $\frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots .\right)$, i. e. less than $\frac{\pi^2}{12}$, which is itself less than $\frac{\pi}{4} + \pi$.

In like manner, if we put $X = \frac{3}{2}\pi$, we obtain

$$\tan^{-1}\frac{3^2}{2.1^2} + \tan^{-1}\frac{3^2}{2.2^2} + \tan^{-1}\frac{3^2}{2.3^2} + \ldots . = \frac{\pi}{4} + \lambda\pi;$$

and (with the same understanding concerning $\tan^{-1}$) $\lambda = 1$; for the series on the left is greater than the one above considered, but less than

$$\tan^{-1}\frac{3^2}{2.1^2} + \tan^{-1}\frac{3^2}{2.2^2} + 3\left\{\tan^{-1}\frac{3^2}{2.3^2} + \tan^{-1}\frac{3^2}{2.6^2} + \tan^{-1}\frac{3^2}{2.9^2} + \ldots .\right\},$$

which is itself less than

$$\frac{\pi}{2} + \frac{\pi}{2} + 3\frac{\pi}{4}, = \frac{3\pi}{4} + \pi, < \frac{\pi}{4} + 2\pi.$$

Likewise

$$\tan^{-1}\frac{5^2}{2.1^2} + \tan^{-1}\frac{5^2}{2.2^2} + \tan^{-1}\frac{5^2}{2.3^2} + \ldots . = \frac{\pi}{4} + \lambda\pi,$$

and $\lambda = 2$ because the series on the left is less than

$$\tan^{-1}\frac{5^2}{2.1^2} + \tan^{-1}\frac{5^2}{2.2^2} + \tan^{-1}\frac{5^2}{2.3^2} + \tan^{-1}\frac{5^2}{2.4^2}$$
$$+ 5\left\{\tan^{-1}\frac{5^2}{2.5^2} + \tan^{-1}\frac{5^2}{2.10^2} + \tan^{-1}\frac{5^2}{2.15^2} + \ldots .\right\},$$
$$< \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + 5\frac{\pi}{4}, = \frac{\pi}{4} + 3\pi.$$

This suggests that when $X = (2k+1)\frac{\pi}{2}$, the value of λ is k, giving the formula

$$\tan^{-1}\frac{(2k+1)^2}{2.1^2} + \tan^{-1}\frac{(2k+1)^2}{2.2^2} + \tan^{-1}\frac{(2k+1)^2}{2.3^2} + \ldots\ldots = \frac{\pi}{4} + k\pi,$$

k being any positive integer, and $\tan^{-1}$ being always taken between 0 and $\frac{\pi}{2}$. But it is evident that the above method will not furnish this result; when $2k+1 \overline{\gtrless} 7$, the superior limit arrived at by it will exceed $\frac{\pi}{4} + k\pi$. But the formula would be proved if we could show that

$$\tan^{-1}\frac{n^2}{2r^2} + \tan^{-1}\frac{n^2}{2r'^2} + \tan^{-1}\frac{n^2}{2(n+r)^2} + \tan^{-1}\frac{n^2}{2(n+r')^2}$$
$$+ \tan^{-1}\frac{n^2}{2(2n+r)^2} + \tan^{-1}\frac{n^2}{2(2n+r')^2} + \ldots\ldots = \pi,$$

where $n = 2k+1$ and $r + r' = n$. This last formula may be more compactly written

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1}\frac{n^2}{2(\nu n + r)^2} = \pi.$$

In point of fact it is easy to prove the more general formula

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1}\frac{\alpha^2}{2(\nu\alpha + \beta)^2} = \pi,$$

α and β being any real quantities. There is evidently no loss of generality in supposing, in the demonstration, that α is positive, and that β is positive and less than α. We have seen above that

$$\tan^{-1}\frac{\tanh Y}{\tan X} \equiv \tan^{-1}\frac{Y}{X} + \sum_{1}^{\infty}{}_{\nu}\left(\tan^{-1}\frac{Y}{X - \nu\pi} + \tan^{-1}\frac{Y}{X + \nu\pi}\right) \pmod{\pi};$$

writing, in this, $\frac{X - \beta\pi}{\alpha}$ for X and $\frac{Y}{\alpha}$ for Y, it becomes

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X-\beta\pi}{\alpha}} \equiv \tan^{-1}\frac{Y}{X - \beta\pi}$$
$$+ \sum_{1}^{\infty}{}_{\nu}\left\{\tan^{-1}\frac{Y}{X - (\nu\alpha + \beta)\pi} + \tan^{-1}\frac{Y}{X + (\nu\alpha - \beta)\pi}\right\} \pmod{\pi}.$$

Likewise

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}\equiv\tan^{-1}\frac{Y}{X+\beta\pi}$$

$$+\sum_{1}^{\infty}{}_{\nu}\left\{\tan^{-1}\frac{Y}{X-(\nu\alpha-\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha+\beta)\pi}\right\}\pmod{\pi}.$$

Hence, by addition,

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X-\beta\pi}{\alpha}}+\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}\equiv\tan^{-1}\frac{Y}{X-\beta\pi}+\tan^{-1}\frac{Y}{X+\beta\pi}$$

$$+\sum_{1}^{\infty}{}_{\nu}\left\{\tan^{-1}\frac{Y}{X-(\nu\alpha+\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha+\beta)\pi}\right.$$

$$\left.+\tan^{-1}\frac{Y}{X-(\nu\alpha-\beta)\pi}+\tan^{-1}\frac{Y}{X+(\nu\alpha-\beta)\pi}\right\}\pmod{\pi},$$

and it is plain that this may be written

$$\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X-\beta\pi}{\alpha}}+\tan^{-1}\frac{\tanh\frac{Y}{\alpha}}{\tan\frac{X+\beta\pi}{\alpha}}$$

$$\equiv\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{2XY}{X^2-Y^2-(\nu\alpha+\beta)^2\pi^2}\pmod{\pi}\quad\text{(C)}$$

Putting, in this identity, $X=Y=\alpha\frac{\pi}{2}$, the first member is $0\pmod{\pi}$ because $\tan\left(\frac{1}{2}-\frac{\beta}{\alpha}\right)\pi=-\tan\left(\frac{1}{2}+\frac{\beta}{\alpha}\right)\pi$; hence

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{\alpha^2}{2(\nu\alpha+\beta)^2}\equiv 0\pmod{\pi}.$$

Finally, to determine the value of the series exactly (each term being taken between 0 and $\frac{\pi}{2}\Big)$ we have only to observe that the sum from $\nu = 1$ to $\nu = +\infty$ can not exceed $\sum_1^\infty {}_\nu \tan^{-1}\frac{1}{2\nu^2}$, which has been proved to be equal to $\frac{1}{4}\pi$; that the sum from $\nu = -2$ to $\nu = -\infty$ is less than $\sum_1^\infty {}_\nu \tan^{-1}\frac{1}{2\nu^2}$; and the sum of the two omitted terms ($\nu = 0$, $\nu = 1$) must be less than twice $\frac{1}{2}\pi$. Hence the proposed series is less than $\frac{1}{4}\pi + \frac{1}{4}\pi + \pi$, and therefore less than 2π. Hence

$$\sum_{-\infty}^{+\infty} {}_\nu \tan^{-1}\frac{\alpha^2}{2(\nu\alpha + \beta)^2} = \pi, \tag{D}$$

α and β being any real quantities; Q. E. D. We have thus completed the proof of the equation

$$\tan^{-1}\frac{(2k+1)^2}{2.1^2} + \tan^{-1}\frac{(2k+1)^2}{2.2^2} + \tan^{-1}\frac{(2k+1)^2}{2.3^2} + \ldots . = \frac{\pi}{4} + k\pi. \tag{E}$$

If, in formula B, we put $X = 2k.\frac{\pi}{2}$, we get

$$\tan^{-1}\frac{(2k)^2}{2.1^2} + \tan^{-1}\frac{(2k)^2}{2.2^2} + \tan^{-1}\frac{(2k)^2}{2.3^2} + \ldots . \equiv -\frac{\pi}{4} \pmod{\pi},$$

and equation D enables us immediately to determine the actual value of the series; viz., putting $\alpha = 2k$, and β successively $= 0, 1, 2, \ldots, 2k - 1$, and adding the results, we get the preceding series doubled, and in addition a term $\frac{\pi}{2}$ (arising from $\beta = 0$, $\nu = 0$); hence

$$\tan^{-1}\frac{(2k)^2}{2.1^2} + \tan^{-1}\frac{(2k)^2}{2.2^2} + \tan^{-1}\frac{(2k)^2}{2.3^2} + \ldots .$$

$$= \frac{1}{2}\left(2k\pi - \frac{1}{2}\pi\right) = -\frac{\pi}{4} + k\pi. \tag{F}$$

Equations E and F may be combined into the single equation

$$\tan^{-1}\frac{n^2}{2.1^2}+\tan^{-1}\frac{n^2}{2.2^2}+\tan^{-1}\frac{n^2}{2.3^2}+\ldots.=\left(\frac{n}{2}-\frac{1}{4}\right)\pi, \tag{G}$$

n being any positive integer.

Putting $\beta/\alpha=u$, equation D may be written more simply

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{1}{2(\nu+u)^2}=\pi, \tag{D$'$}$$

u being any real quantity. It seems very remarkable that the value of this sum should be independent of u. And if, in formula C, we put $X=Y=n\alpha\frac{\pi}{2}$, n being any positive integer, we obtain

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{n^2}{2(\nu+u)^2}\equiv 0 \pmod{\pi}.$$

The actual value of the series is $n\pi$. For the sum from $\nu=1$ to $\nu=+\infty$ does not exceed $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$* (by equation G), the sum from $\nu=-2$ to $\nu=-\infty$ is less than $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$, and the two omitted terms are together less than π; and on the other hand the sum from $\nu=0$ to $\nu=+\infty$ is at least as great as $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$, and the sum from $\nu=-1$ to $\nu=-\infty$ is at least as great as $\left(\frac{n}{2}-\frac{1}{4}\right)\pi$. Hence the value of the series lies between $\left(n-\frac{1}{2}\right)\pi$ and $\left(n+\frac{1}{2}\right)\pi$; but it is an exact multiple of π; therefore

$$\sum_{-\infty}^{+\infty}{}_{\nu}\tan^{-1}\frac{n^2}{2(\nu+u)^2}=n\pi, \tag{H}$$

n being any positive integer. This is a generalization of equation D or D$'$; but also equation G, which was employed in the proof of H, is itself an obvious corollary from H, so that this last equation may be said to comprise all the preceding arithmetical equations $(D - G)$.

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*In the reasoning, u is supposed non-negative; obviously the argument is precisely similar if u is supposed non-positive.

Finally, we remark that formula C shows that the series

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1} \frac{a}{(\nu + u)^2}$$

is not independent of u except when a is half the square of an integer;* and it thus seems all the more remarkable that it should be independent of u in that case. The value of this series (obtained by putting $X = Y = \sqrt{2a}\,\alpha \frac{\pi}{2}$ in formula C) is given by

$$\sum_{-\infty}^{+\infty}{}_{\nu} \tan^{-1} \frac{a}{(\nu + u)^2}$$

$$= - \tan^{-1} \frac{\tanh \sqrt{2a}\,\frac{\pi}{2}}{\tan\left(\sqrt{2a} - 2u\right)\frac{\pi}{2}} - \tan^{-1} \frac{\tanh \sqrt{2a}\,\frac{\pi}{2}}{\tan\left(\sqrt{2a} + 2u\right)\frac{\pi}{2}} + \lambda\pi,$$

λ being an integer so chosen that the result shall lie between $E(\sqrt{2a}).\pi$ and $[E(\sqrt{2a}) + 1]\,\pi$, since the series is evidently intermediate in value between those which would arise on writing instead of a the two half-squares between which it lies.

The orthogonal trajectories of the preceding system of curves are given by the equation

$$\sin x\, dx + \sin y\, dy = 0;$$

these curves have the property

$$- \vartheta_N = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \ldots\,,$$

and their equation is

$$\cos x + \cos y = c \quad \text{or} \quad \cos X \cosh Y = C.$$

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* It is, of course, periodic in respect to u in any case, the period being 1.

The curves having the property

$$\vartheta_T = \vartheta_0 + \vartheta_1 + \vartheta_{-1} + \vartheta_2 + \vartheta_{-2} + \vartheta_3 + \vartheta_{-3} + \cdots\cdot$$

are given by the differential equation

$$dx/\sin x = dy/\sin y;$$

the solution of this is

$$\sin\frac{1}{2}(x+y) = c\sin\frac{1}{2}(x-y) \quad \text{or} \quad \sin X = C\sinh Y.$$

This last equation is otherwise evident from the consideration that the curve now required is so defined that its angle of intersection with any curve of the system first considered in this section shall be bisected by a parallel to the axis of X. Hence, the equation of that system having been found to be $\sin X \sinh Y = C$, it is plain that the required curves are given by $\sin X = C\sinh Y$.